Classical Demand Theory II

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We have covered the following in the last lecture:

- Overview
- Preference Relations
- Preference and Utility
- The Utility Maximization Problem
- The Expenditure Minimization Problem

We will cover these today:

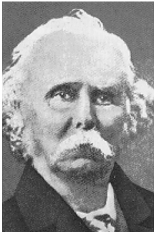
- Relationship between Demand, Indirect Utility and Expenditure Functions
- The Strong Axiom of Revealed Preference
- *Aggregation
- *Ceteris Paribus and the Law of Demand

Note: * indicates topics outside Ch.3. Also, we will not cover "Duality" and "Integrability". "Welfare" might be postponed later.

- We've been talking about the Walrasian Demand Function. At times, it is called the Marshallian Demand Function or the Ordinary Demand Function.
- MWG, however, thinks it's inappropriate to call the Walrasian Demand Function Marshallian because Marshall used partial equilibrium analysis in which wealth effects are absent so all demand functions discussed in this chapter are the same.

Alfred Marshall (1842-1924)

- In *Principles of Economics*, he popularized that supply and demand determine prices. This was also discovered by Cournot
- He invented the concepts of marginal analysis, price elasticity of demand, Giffen goods, consumer and producer supluses
- He limited his quantitative expressions so that he might appeal to the layman
- He wrote to A.C. Pigou: "(1) Use mathematics as shorthand language, rather than as an engine of inquiry. (2) Keep to them till you have done. (3) Translate into English. (4) Then illustrate by examples that are important in real life (5) Burn the mathematics. (6) If you can't succeed in 4, burn 3. This I do often."



Milton Friedman (1912-2006)

- “for his achievements in the fields of consumption analysis, monetary history and theory, and for his demonstration of the complexity of stabilization policy.” Press Release on 14 October 1976, The Royal Swedish Academy of Sciences.
- His paper “The Marshallian Demand Curve,” *Journal of Political Economy* (Dec 1949) is perhaps the best paper ever written on demand theory.

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Hicksian Demand and the Expenditure Function

Proposition 3.G.1: Suppose that the continuous $u(\cdot)$ represents a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. We have:

$$\forall l, p, u : h_l(p, u) = \frac{\partial e(p, u)}{\partial p_l}, \text{or} \tag{1}$$

$$h(p, u) = \nabla_p e(p, u) \tag{2}$$

Proof of Proposition 3.G.1

Proof 2: (First-Order Conditions Argument)

- For simplicity, assume $h(p,u) >> 0$, so we have an interior solution.
- $$\begin{aligned}
\nabla_p e(p,u) &= \nabla_p [p \cdot h(p,u)] && (3)\\
&= h(p,u) + [p \cdot D_p h(p,u)]^T \text{ (by the chain rule)} && (4)\\
&\uparrow \text{(use also the fact that } \frac{\partial b^T a}{\partial a} = \frac{\partial a^T b}{\partial a} = b^T\text{)} && (5)\\
&= h(p,u) + [\lambda \nabla u(h(p,u) \cdot D_p h(p,u)]^T && (6)\\
&\uparrow \text{(by the FOC of an interior solution)} && (7)\\
&= h(p,u) + 0 \text{(since } u(h(p,u)) = u \text{ holds for all } p\text{)} && (8)
\end{aligned}$$

Note: We omit Proof 1: (Duality Argument). Please read Proof 3: (Envelope Theorem Argument) yourself.

Hicksian Demand and the Expenditure Function

Proposition 3.G.2: Suppose that the continuous $u(\cdot)$ represents a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. We have:

1. $D_p h(p, u) = D_p^2 e(p, u)$
2. $D_p h(p, u)$ is a negative semidefinite matrix
3. $D_p h(p, u)$ is a symmetric matrix
4. $D_p h(p, u) p = 0$

Comments

Bullet point 2 (Negative Semidefiniteness) is the differential analog of the compensated law of demand because:

- $D_p h(p, u)$ is a negative semidefinite matrix
  - $\implies dp \cdot D_p h(p, u) dp \leq 0$
  - $\implies dp \cdot dh(p, u) \leq 0$ since $dh(p, u) = D_p h(p, u) dp$
- Negative semidefiniteness implies that $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$ for all l.

Comments

Bullet point 3 (Symmetry) implies path independence.

- $\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial h_k(p,u)}{\partial p_l}$
- Intuitively, when you climb a mountain, you will cover the same height regardless of the route
- Closely linked to the transitivity of rational preference (more in Ch 13)

Comments

Bullet point 4 $(D_p h(p,u)p = 0)$ implies that each good has at least one substitute

- By negative semidefinitess, $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$ for all l.
- Inner product $= 0 \implies \frac{\partial h_l(p,u)}{\partial p_k} \geq 0$ for some k,for all l.
  - Definitions: Substitutes: $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$, Complements: $\frac{\partial h_l(p,u)}{\partial p_k} \leq 0$, Gross Substitutes: $\frac{\partial x_l(p,w)}{\partial p_k} \geq 0$, Gross Complements: $\frac{\partial x_l(p,w)}{\partial p_k} \leq 0$,

Hicksian and Walrasian Demand Functions

- MWG says:
    - $h(p, u)$ is not observable
    - $x(p, w)$ is observable
    - But we can link them using the Slutsky Equation so that we can derive the unobservable from the observable

- $x(p, w)$ is observable.....really? It's a huge error.

Example (Cheung 2003, Book 1, Chapter 4)

Unit of Apples	A's MV	B's MV
1	\$1	\$2
2	\$0.9	\$1.6
3	\$0.8	\$1.2
4	\$0.7	\$0.8
5	\$0.6	\$0.4
6	\$0.5	\$0.0

- If each person can be a seller or a buyer, the equilibrium price is \$0.8
- The transacted volume is observable. And the number of units sold equals the number of units purchased, which is 4.
- But both the quantity demanded and quantity supplied are 6 at \$0.8.

- For now, just assume that $x(p, w)$ is observable in order to get through the arguments in MWG.

Proposition 3.G.3: (The Slutsky Equation) Suppose that the continuous $u(\cdot)$ represents a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}_+^L$. We have:

$$\forall l, k : \frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w) \qquad (9)$$

Proof: Verify it yourself.

Substitution and Wealth Effects

$$\forall l, k : \underbrace{\frac{\partial x_l(p,w)}{\partial p_k} \triangle p_k}_{\text{Total Effect}} \approx \underbrace{\frac{\partial h_l(p,u)}{\partial p_k} \triangle p_k}_{\text{Substitution Effect}} \underbrace{- \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \triangle p_k}_{\text{Wealth Effect}} \tag{10}$$

Sometimes we instead say:

$$\forall l, k : \underbrace{\frac{\partial x_l(p,w)}{\partial p_k}}_{\text{Total Effect}} \triangle p_k \approx \underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{\text{Substitution Effect}} \triangle p_k \underbrace{- \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{\text{Wealth Effect}} \triangle p_k \tag{11}$$

Slutsky Substitution Matrix

- Recall that we define in Ch 2 the Slutsky Substitution Matrix:
$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \dots & s_{1L}(p,w) \\ & \ddots & \\ s_{L1}(p,w) & \dots & s_{LL}(p,w) \end{bmatrix}.$$
- By definition of $S(p,w)$, it turned out that $\frac{\partial h_l(p,u)}{\partial p_k} = s_{lk}$.

Verify yourself the relationship between Hicksian and Walrasian demand with normal and inferior goods. See Fig 3.G.1.

- Suppose we are initially at $\bar{x} = x(\bar{p}, \bar{w})$ and $\bar{u} = u(\bar{x})$, and we want to change wealth to compensate for wealth effect arising from a price change. There are two ways:
- Slutsky Wealth Compensation
  - $\triangle w_{Slutsky} = p' \cdot x(\bar{p}, \bar{w}) - \bar{w}$
  - then, the consumer is just able to afford his initial bundle
- Hicksian Wealth Compensation
  - $\triangle w_{Hicks} = e(p', \bar{u}) - \bar{w}$
  - then, the consumer's utility is unchanged
- Discrete change (say $\triangle p_1$): $\triangle w_{Hicks} < \triangle w_{Slutsky}$
  - Homework: Prove that $\triangle w_{Hicks} < \triangle w_{Slutsky}$ for discrete change in price. See Fig 3.G.2 for geometrical intuition.
- Differential change: $\triangle w_{Slutsky} = \triangle w_{Hicks}$ because by Prop 3.G.1 $\nabla_p e(\bar{p}, \bar{u}) = h(\bar{p}, \bar{u})$, so:
  - $e(p', \bar{u}) = p' \cdot h(p', \bar{u}) = p' \cdot x(p', e(p', \bar{u})) = p' \cdot x(\bar{p}, \bar{w})$
  - Homework: Why the last equality holds?

Walrasian Demand and the Indirect Utility Function

- We know from Prop 3.G.1 $h(p, u) = \nabla_p e(p, u)$
- But this is not true: $x(p, w) = \nabla_p v(p, w)$ because:
  - Homework: $x(p, w)$ is an ordinal concept while $v(p, w)$ is not. Why? And why is this connected to $x(p, w) \neq \nabla_p v(p, w)$?
- However, thru the Roy's Identity, it's true if we normalize the RHS by the marginal utility of wealth.
- Proposition 3.G.4 (Roy's Identity): Suppose that the continuous $u(\cdot)$ represents a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}$ and $(\bar{p}, \bar{w}) >> 0$. We have:

$$x(\bar{p}, \bar{w}) = \frac{-1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w}) \tag{12}$$

Proof 1 of Roy's Identity:

- Let $\bar{u} = v(\bar{p}, \bar{w})$. Since $v(p, e(p, \bar{u})) = \bar{u}$ holds for all p, we can differentiate it wrt p and evaluate it at $\bar{p}$:
- $$\nabla_p v(\bar{p}, e(\bar{p}, \bar{u})) + \frac{\partial v(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \nabla_p e(\bar{p}, \bar{u}) = 0 \qquad (13)$$
- Then substitute $\nabla_p e(\bar{p}, \bar{u}) = h(\bar{p}, \bar{u})$ (by Prop. 3.G.1), $\bar{w} = e(\bar{p}, \bar{u})$ and $h(\bar{p}, \bar{u}) = x(\bar{p}, \bar{w})$. Q.E.D.
- Note: Go through Proof 2 (using FOC) and Proof 3 (using Envelope Theorem) yourself.

Relationship between Demand, Indirect Utility and Expenditure Functions

→The Strong Axiom of Revealed Preference

Aggregation

Ceteris Paribus and the Law of Demand

- In Ch. 2, we saw that we cannot directly apply some results in Ch. 1 between the Weak Axiom and rational preference because:
  - Recall that Proposition 1.D.2 says If $\mathcal{B}$, a set of some nonempty subsets of the set of alternatives X, includes all subsets of X of up to three elements, then there is a unique $\succsim$ that "rationalizes" the choice rule.
  - But the Walrasian budget sets does not include all possible subsets of the 2- or 3-commodity bundles
- In Homework MWG Ex. 2.F.2, you were asked to show Hicks' (1956) result that the satisfaction of the WA does not imply that the preference is rational.
- We now present a result: the satisfaction of the Strong Axiom (SA) does imply that the preference is rational.

- The SA says that if $x(p^1, w^1)$ is directly or indirectly revealed preferred to $x(p^N, w^N)$, then $x(p^N, w^N)$ cannot be directly revealed preferred to $x(p^1, w^1)$ [so $x(p^1, w^1)$ cannot be affordable at (p^N, w^N)].
- The directness is in the same sense of the WA. But the indirectness means some form of transitivity: say, if $N = 3$, when $x(p^2, w^2)$ and $x(p^3, w^3)$ are both affordable at (p^2, w^2) and since $x(p^3, w^3)$ is not chosen at $(p^2, w^2), x(p^2, w^2)$ is directly revealed preferred to $x(p^3, w^3)$. Similarly $x(p^1, w^1)$ is directly revealed preferred to $x(p^2, w^2)$. So the SA wants to impose that $x(p^1, w^1)$ must be indirectly revealed preferred to $x(p^3, w^3)$ and thus $x(p^1, w^1)$ cannot be affordable at (p^3, w^3).

- The general version is in Definition 3.J.1: $x(p, w)$ satisfies the SA if for any list:

$$(p^1, w^1), ..., (p^N, w^N) \tag{14}$$

  with $(p^{n+1}, w^{n+1}) \neq (p^n, w^n)$ for all $n \leq N-1$, we have $p^N \cdot x(p^1, w^1) > w^N$ whenever $p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \leq N-1$.
- Comments: Set $N = 2$, SA is equivalent to WA. Homework Ex 3.J.1 When $L = 2$ (not necessarily $N = 2$), SA is equivalent to WA.

Examples

- Verify yourself that Example 2.F.1 violates the SA.

Proposition 3.J.1 $x(p, w)$ satisfies the SA $\implies \exists$ a rational $\succsim$ that rationalizes $x\left(p, w\right)$. That is, $\forall(p, w), y \neq x\left(p, w\right)$ and $y \in B_{p,w} : x\left(p, w\right) \succ y$

Comments:

- This result is due to Houthakker (1950).
- The proof presented in MWG is by Richter (1966), which involves set theory, is a bit advanced to be proven here.
- Note that the converse is straightforward because rationality implies transitivity.
- SA also requires symmetry of the Slutsky Matrix (from a result in Ch. 13) implying path independence, and negative semidefiniteness (since the WA must satisfy this by Proposition 3.G.2)

Relationship between Demand, Indirect Utility and Expenditure Functions

The Strong Axiom of Revealed Preference

→Aggregation

Ceteris Paribus and the Law of Demand

- Definition: The market demand is the summation of the Walrasian demand function of each of the I individual:

$$x(p, w_1, ..., w_I) = \sum_{i=1}^{I} x_i(p, w_i) \tag{15}$$

- A central question: Since oftentimes we can only observe aggregate wealth and "observe" demand, can we write this?

$$\sum_{i=1}^{I} x_i(p, w_i) = x_i(p, \sum_{i=1}^{I} w_i) \tag{16}$$

- This implies that the effect due to the statistical distribution of wealth across individuals is absent. This absence can be interpreted as this: for the initial distribution $(w_1, ..., w_I)$ and a differential change in wealth $(dw_1, ..., dw_I) \in \mathbb{R}^I$ such that $\sum_i dw_i = 0$, we have:

$$\sum_{i=1}^{I} \frac{\partial x_{li}(p, w_i)}{\partial w_i} dw_i = 0 \tag{17}$$

- But for this to happen, the necessary and sufficient condition is that the wealth expansion paths are the same for different individuals:

$$\frac{\partial x_{li}(p, w_i)}{\partial w_i} = \frac{\partial x_{lj}(p, w_i)}{\partial w_j} \tag{18}$$

- Comments: Verify yourself that when consumers have the same homothetic preferences or that all consumers have preferences that are quasilinear to the same good, then Proposition 4.B.1 holds.
- More generally, whenever the indirect utility function is in the Gorman form, then $\sum_{i=1}^{I} x_i(p, w_i) = x(p, \sum_{i=1}^{I} w_i)$:
- Proposition 4.B.1

$$\sum_{i=1}^{I} x_i(p, w_i) = x(p, \sum_{i=1}^{I} w_i) \iff \tag{19}$$

$$\forall i : v_i(p, w_i) = a_i(p) + b(p) w_i \tag{20}$$

- Proof: Sufficiency, use the Roy's identity (Homework Ex. 4.B.1). Necessity, too advanced to be proven here (see Deaton and Muellbauer (1980)).

Aggregate Demand and the Weak Axiom

- Suppose Proposition 4.B.1 holds. Specifically, consider $\alpha_i \geq 0, \sum_i \alpha_i = 1$ so that $w_i = \alpha_i w$.
- Now the aggregate demand does not satisfy the WA when $\alpha_i = 1/2$ and $I = 2$.
  - Proof: See Example 4.C.1 and Figure 4.C.2.
- The failure is due to wealth effects because Proposition 2.F.1 tells us that the WA holds iff it satisfies the compensated law of demand.
- It can be easily shown that if each individual has a compensated price change, then the aggregate demand satisfies the WA. Even if some individual has a uncompensated price change:
- Proposition 4.C.1 As long as each individual satisfies the uncompensated law of demand (ULD), so does the aggregate and in turn the aggregate demand satisfies the WA.
- Proof: Verify it yourself (See MWG p.112.)

- Comments:
  - Definition 4.C.2 An individual satisfies the uncompensated law of demand (ULD) if

    $$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0 \qquad (21)$$

    for all p', p, w_i, with strict inequality if $x_i(p', w_i) \neq x_i(p, w_i)$.
  - ULD is not implied by preference maximization (MWG Ex. 4.C.3)
  - But if $\succsim_i$is homothetic (Proposition 4.C.2) or the substitution effects relative to the wealth effects are large enough $(\frac{-x_i \cdot D^2 u_i(x_i) x_i}{x_i \cdot \nabla u_i(x_i)} < 4)$ (Proposition 4.C.3), then ULD holds.
  - It is not necessary for each individual to satisfy the ULD for the aggregate to satisfy the ULD (Proposition 4.C.4 gives an example).
  - Proposition 4.C.2 to 4.C.4 and Section 4.D are out of our syllabus

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→Ceteris Paribus and the Law of Demand

- Oftentimes, the compensated law of demand requires that ceteris paribus (other things being equal), when price changes, quantity demanded moves in the opposite direction
- But what induce a change in price if all other things being equal? What do you think?

- Friedmanian (1949) Demand Function [Friedmanian (1949)] investigate the Marshallian Demand Function in a general equilibrium setting
- He said it doesn't matter if we are holding nominal or real income fixed, under full employment, the effects are the same and no Giffen Goods will result since a price change only change resource allocation but not changes in the real wealth. That is to say, Giffen goods only happened in a partial equilibrium setting
- Cheung (2003) thinks that it's not right because there are exogenous factors (such as a sudden increase in crop yields) will increase society's real wealth. He suggests another way to resolve the Giffen paradox.

Example (Cheung 2003, Book 1, Chapter 4) Revisited

Unit of Apples	A's MV	B's MV
1	\$1	\$2
2	\$0.9	\$1.6
3	\$0.8	\$1.2
4	\$0.7	\$0.8
5	\$0.6	\$0.4
6	\$0.5	\$0.0

- Cheung claimed without proof that Giffen goods can't exist in the market because else there will be no transactions. Agree?

What factors are allowed to change? (Cheung 2003)

- Allowed:
  - Factors that directly affect prices
  - Factors that first affect prices then quantity demanded, or vice versa.
    - Reason: Too complicated to take into account all these
- Disallowed:
  - Factors that directly shift the demand curve

But do we need to care about finding out what factors have been changed?

- (Cheung 2003): No. As long as we can identify the constraints that are pivotal so that the quantity demanded moved in the same direction as the quantity purchased