Consumers

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- Economists disagree with the use of utility or preference functions, especially because they are not observable
- This lecture develops a few fundamental results in economics without an explicit formulation of such functions except the use of revealed preference
- Next lectures we will make the formulation explicit

John R Hicks

- "In his most well-known work...Value and Capital,...He presented a complete economic equilibrium model with aggregated markets for commodities, factors of production, credit and money. The construction of this model included a number of innovations, i.e., a further development of older theories of consumption and of production..." Press Release on 25 October 1972, The Royal Swedish Academy of Sciences.

Paul A. Samuelson

- "for the scientific work through which he has developed static and dynamic economic theory and actively contributed to raising the level of analysis in economic science..." The Royal Swedish Academy of Sciences, 1970.

Overview
$\rightarrow$Demand Correspondence
Tools Associated with Demand Correspondence
Types of Goods
Weak Axiom of Revealed Preference
WA with Compensated Price Change
Slutsky Matrix

Setting

- L commodities
- Commodity bundle (vector): $x = (x_1, ..., x_L) \in \mathbb{R}^L$
- Consumption set: $X \subseteq \mathbb{R}^L$ is the set of all feasible commodity bundles
- Price vector vector: $p = (p_1, ..., p_L) \in \mathbb{R}^L$
- Unless stated otherwise, we only consider $X = \mathbb{R}^L_+$ and $p \in \mathbb{R}^L_{++}$

- The Walrasian or competitive budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is the set of all feasible consumption bundles for the consumer who takes market prices as given and has wealth w.
- $\{x \in \mathbb{R}^L_+ : p \cdot x = w\}$ is the budget hyperplane (or line if $L = 2$)
- See figures 2.D.1 to 2.D.3
- In-class exercise: Show that $B_{p,w}$ is a convex set
- Ex. (see Ex. 2.D.3): Show that, in general, $B_{p,w}$ is convex if X is convex. Is the converse true?
- Other consumption sets (see Fig. 2.D.4)

- $x(p, w)$ is the demand correspondence, which assigns a set of chosen consumption bundles for each price-wealth pair (p, w).
  - It's called the demand function if it's a singleton
  - Write $x(p, w)$ as $[x_1(p, w), x_2(p, w), ..., x_L(p, w)]$

- Definition 2.E.1. $x(p, w)$ is homogeneous of degree zero iff $x(p, w) = x(\alpha p, \alpha w), \forall p, w, \alpha > 0$.
  - Note: Since $B_{p,w} = B_{\alpha p, \alpha w}$, then homogeneity of degree zero says that consumption choice depends only on the budget set when price and wealth are multiplied by the same amount
  - In-class exercise: How many variables will $x(p, w)$ depend on when we set $\alpha = \frac{1}{p_1}$ or $\frac{1}{w}$?

- Definition 2.E.2 $x(p, w)$ satisfies Walras' law iff $\forall p >> 0, w > 0, p \cdot x = w, \forall x \in x(p, w)$.
  - Note: It says that the consumer fully expends his wealth
  - Wealth in economics refers to the net present value of future income stream. If you have a constant income (I) every period with a discount rate of r :

$$w = I + \frac{I}{(1+r)} + \frac{I}{(1+r)^2} + ... \quad (1)$$

$$= I(\frac{1}{1 - \frac{1}{(1+r)}}) = \frac{I}{r} \quad (2)$$

  - The above two definitions will rather be results not definitions when we derive them from utility maximization under some general circumstances in Ch. 3.
  - w and I are nominal variables. When adjusted for price level, they become real variables.

Technical Assumptions

When appropriate, we will assume that $x(p, w)$ is single-valued, continuous and differentiable.

- A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at c iff $\forall \epsilon > 0, \exists \delta \; s.t. |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
- A function f is continuous if it's continuous at every point
- Fact: It can be shown that a function is differentiable at c implies that it is continuous at c. The converse is not true. Verify that the absolute value function is continuous but not differentiable at zero.

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- Engel function: The function of wealth $x(\bar{p}, w)$ for fixed $\bar{p}$.
- Wealth (Income) expansion path: $E_{\bar{p}} = \{x(\bar{p}, w) : w > 0\}$
- Wealth (Income) effect for the l^{th} good: $\partial x_l(p, w)/\partial w$
- Wealth (Income) effects: $D_w x(p, w) = [\partial x_1(p, w)/\partial w, \partial x_2(p, w)/\partial w, ..., \partial x_L(p, w)/\partial w]$
- See Figure 2.E.1

- Price effect of p_k on x_l: $\partial x_l(p, w)/\partial p_k$
- Price effects:

$$D_p x(p, w) = \begin{bmatrix} \partial x_1(p, w)/\partial p_1 & \dots & \partial x_1(p, w)/\partial p_L \\ & \ddots & \\ \partial x_L(p, w)/\partial p_1 & \dots & \partial x_L(p, w)/\partial p_L \end{bmatrix}$$

- Offer curve: $O_{\bar{w}, \bar{p}_{-l}} = \{x((\bar{p}_{-l}, p_l), \bar{w}) : \bar{w} > 0\}$

- Relationship between price effects and changes in **real** wealth
- See MWG Figure 2.E.3.

- Proposition 2.E.1 If $x(p, w)$ is homogeneous of degree zero, then:

$$\forall p, w : \sum_{k=1}^{L} \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0, l = 1, ..., L \quad (3)$$

- It says that the price and wealth effects for any good l, when weighted by these prices and wealth, sum to zero.
- For easier interpretation, use elasticities.

- Elasticities are unit-free measure of change responsiveness
- Elasticities of demand wrt to price:

$$\epsilon_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k}\frac{p_k}{x_l(p,w)} \tag{4}$$

  - Think of it as: $\frac{\triangle x_l / x_l}{\triangle p_k / p_k}$, the % change of x as a response to the % change of p_k
  - Can be approximated by $\epsilon_{lk} = \frac{(x_l^2 - x_l^1)/(0.5)(x_l^2 + x_l^1)}{(p_k^2 - p_k^1)/(0.5)(p_k^2 - p_k^1)}$
- Elasticities of demand wrt to wealth:

$$\epsilon_{lw}(p,w) = \frac{\partial x_l(p,w)}{\partial w}\frac{w}{x_l(p,w)} \tag{5}$$

- Divide both sides of Proposition 2.E.1 by $x_l(p,w)$:

$$\forall p,w: \sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k}\frac{p_k}{x_l(p,w)} + \frac{\partial x_l(p,w)}{\partial w}\frac{w}{x_l(p,w)} = 0, l = 1, ..., L \tag{6}$$

which is equivalent to:

$$\forall p,w: \sum_{k=1}^{L} \epsilon_{lk}(p,w) + \epsilon_{lw}(p,w) = 0, l = 1, ..., L \tag{7}$$

- Interpretation: An equal percentage change in all prices and wealth leads to no change in demand, which is consistent with what the homogeneity of degree zero tells us.

Proof of Proposition 2.E.1

- Recall that Proposition 2.E.1 says if $x(p, w)$ is homogeneous of degree zero, then:

$$\forall p, w : \sum_{k=1}^{L} \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0, l = 1, ..., L \quad (8)$$

- Proof: By homogeneity, $\forall \alpha > 0 : x_l(\alpha p, \alpha w) - x_l(p, w) = 0$. Take derivative wrt α, by the chain rule, we have:

$$\sum_{k=1}^{L} \frac{\partial x_l(\alpha p, \alpha w)}{\partial \alpha p_k} \frac{\partial \alpha p_k}{\partial \alpha} + \frac{\partial x_l(\alpha p, \alpha w)}{\partial \alpha w} \frac{\partial \alpha w}{\partial \alpha} = 0 \quad (9)$$

Then set $\alpha = 1$.

Relationship with the Firm's Problem

- Consider a good X and a price level P
- $MR = \frac{d(\text{Total Revenue})}{dX} = \frac{dPX}{dX} = P\frac{dX}{dX} + X\frac{dP}{dX} = P(1 + \frac{X}{dX}\frac{dP}{P}) = P(1 + \frac{1}{\epsilon})$
- ϵ is typically negative (by the law of demand we will see in this and next lectures)
- If $\epsilon < -1$, demand is elastic, MR is positive
- If $-1 < \epsilon < 0$, demand is inelastic, MR is negative
- If $\epsilon = -1$, demand is unitary, MR is zero

Cournot Aggregation

- Proposition 2.E.2: If $x(p, w)$ satifies the Walras' law, then:

$$\forall p, w : \sum_{k=1}^{L} p_l \frac{\partial x_l(p, w)}{\partial p_k} + x_k(p, w) = 0, k = 1, ..., L \quad (10)$$

  - Proof: Differentiate $p \cdot x(p, w) = w$ w.r.t. $p_k, k = 1, ..., L$.
- It says that the price effects wrt p_k, when weighted by these prices and the demand for good k, sum to zero.
- Multiply both sides by dp_k
  - Interpretation: The change in total expenditure $(|\sum_{k=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} dp_k|)$ must equal the change in wealth $(|x_k(p, w) dp_k|)$
  - For another interpretation, use elasticities

- Proposition 2.E.2 can be interpreted through elasticities (by way of Homework MWG Ex. 2.E.2):

$$\forall p, w : \sum_{k=1}^{L} b_l(p, w)\epsilon_{lk}(p, w) + b_k(p, w) = 0, l = 1, ..., L, \quad (11)$$

  - where $b_l(p, w) = p_l x_l(p, w)/w$ is the budget share of the consumer's expenditure on good l given p and w.

Engel Aggregation

- Proposition 2.E.3: If $x(p, w)$ satifies the Walras' law, then:

$$\forall p, w : \sum_{k=1}^{L} p_l \frac{\partial x_l(p, w)}{\partial w} = 1 \tag{12}$$

  - Proof: Differentiate $p \cdot x(p, w) = w$ w.r.t. w.
- Multiply both sides by dw
  - Interpretation: The change in total expenditure ($|\sum_{k=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} dw|$) must equal to any wealth change ($|dw|$)
  - For another interpretation, use elasticities

- Proposition 2.E.3 can be interpretated through elasticities (by way of Homework MWG Ex. 2.E.2):

$$\forall p, w : \sum_{k=1}^{L} b_l(p, w)\epsilon_{lw}(p, w) = 1, l = 1, ..., L, \tag{13}$$

  - where $b_l(p, w) = p_l x_l(p, w)/w$ is the budget share of the consumer's expenditure on good l given p and w.

- A commodity l is normal at (p, w) if $\partial x_l(\bar{p}, w)/\partial w \geq 0$.
- A commodity l is inferior at (p, w) if $\partial x_l(\bar{p}, w)/\partial w < 0$
- If it's normal (inferior) everywhere, it's a normal (an inferior) commodity
- Examples?

- Giffen Goods at $(p, w) : \partial x_l(\bar{p}, w)/\partial p_l > 0$
- Example: Potatoes? Gasoline?
- MWG: "[Giffen Goods] are not an economic impossibility". Really? Are we allowed to by the method of science?
- More on Giffen Goods in this and next lectures
  - Relationships between Giffen Goods and Inferior Goods
  - Other Issues

WA

- Definition 2.F.1: $x(p, w)$ satisfies the weak axiom of revealed preference if this holds for an abitrary (p, w) and (p', w') :

$$\text{If } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w), \text{ then } p' \cdot x(p, w) > w' \tag{14}$$

- In-class exercises: see graphs

WA with (Slutsky) compensated price change

- For $L = 2$, when there is a change in price, the budget set changes shape as well as volume. That is, the relative price of commodities changes as well as real wealth
- To control for the change in real wealth, we could perform a (positive or negative) nominal wealth adjustment.
- When p is changed to p', the original bundle $x(p, w)$ will still be just affordable when we change the nominal wealth to $w' = p' \cdot x(p, w)$.

WA with (Slutsky) compensated price change

- Proposition 2.F.1: Suppose $x(p, w)$ is homogeneous of degree zero and satisfies the Walras law. Then $x(p, w)$ satisfies the weak axiom of revelated preference iff the compensated law of demand holds.
  - The compensated law of demand says that $\triangle p \cdot \triangle x \leq 0$, or
  - for all compensated price change from (p, w) to $(p', w' = p'x(p, w))$, we have:

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0 \tag{15}$$

    with strict inequality whenever $x(p', w') \neq x(p, w)$.
  - See Figure 2.F.4, 2.F.5

- Proof: " $\Longrightarrow$ " direction. Rewrite LHS as

$$p' \cdot [x(p', w') - x(p, w)] - p \cdot [x(p', w') - x(p, w)] \quad (16)$$

The first term is zero because $w' = p'x(p, w)$. For the second term, by Definition 2.F.1, since $x(p', w')$ is revealed preferred to $x(p, w)$ at $(p', w'), x(p', w')$ must be infeasible at (p, w) (i.e., $p \cdot x(p', w') > w$). Therefore the second term is ≤ 0.

- Proof: " $\Longleftarrow$ " direction.
  - Lemma 1. WA holds if it holds for all compensated price changes
  - Suppose otherwise that WA does not hold. Then by Lemma 1, there exists a compensated price change in which WA does not hold. That is, for a compensated price change from some (p', w') to (p, w) (not the other way round) such that $x(p', w') \neq x(p, w)$ and $p'x(p, w) \leq w'$ we have $p \cdot x(p', w') = w$ (compensation rules out $p \cdot x(p', w') < w$; the WA violation rules out the other direction).
  - By the Walras Law, the last two inequalities imply that

$$p \cdot [x(p', w') - x(p, w)] = 0 \text{ and } p' \cdot [x(p', w') - x(p, w)] \geq 0 \tag{17}$$

  - Substracting:

$$(p' - p) \cdot [x(p', w') - x(p, w)] \geq 0$$

violating $(p' - p) \cdot [x(p', w') - x(p, w)] < 0$ in Proposition 2.F.1.

Lemma. WA holds if it holds for all compensated price changes
Proof: To prove $A \Longleftarrow B$, it is equivalent to proving $\neg A \implies \neg B$. Suppose WA doesn't hold, show that there exists a compensated price change that WA doesn't hold. Violation of WA implies that $\exists (p'', w''), (p', w')$:

$$p' \cdot x(p'', w'') \leq w' \text{ and } x(p'', w'') \neq x(p', w'), \text{ then } p'' x(p', w') \leq w'' \tag{18}$$

We just need to show that one of these inequalities holds with equality because this implies it is a compensated price change. Suppose otherwise that both inequalities do not hold with equality, then construct a third budget set $B_{p,w}$ such that both $x(p'', w'')$ and $x(p', w')$ are on the budget line.

Since they are on the new budget line, we have $p \cdot x(p'', w'') = w$ and $p \cdot x(p', w') = w$. Then by the Walras' law and the second inequality not holding with equality, for all $\alpha \in (0, 1)$:

$$\alpha p' x(p', w') + (1 - \alpha) p'' x(p', w') < \alpha w' + (1 - \alpha) w'' \qquad (19)$$

Rearranging the LHS, it is equal to $(\alpha p' + (1 - \alpha) p'') x(p', w')$. Choose α s.t. $\alpha p' + (1 - \alpha) p'' = p$ and s.t. the LHS becomes w, which equals $p \cdot x(p, w) = \alpha p' x(p, w) + (1 - \alpha) p'' x(p, w)$. Therefore, either $p' x(p, w) < w'$ or $p'' x(p, w) < w''$. Suppose that the first possibility holds (the argument is the same if it's the second possibility). Then we have $x(p', w') \neq x(p, w)$, $p \cdot x(p', w') = w$ and $p' x(p, w) < w'$, which violates the WA under this compensated price change.

Comments on the proof on MWG
1. It doesn't prove WA holds only if it holds for all compensated price changes, which isn't needed in the proof of Proposition 2.F.1. However, the proof is trivial.

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Issue

- When the price of a good decreases, real wealth increases. How do we conceptualize the change of consumption due to the price change but not real wealth change?

Slutsky Substitution Matrix

- $S(p, w) = \begin{bmatrix} s_{11}(p, w) & \dots & s_{1L}(p, w) \\ & \ddots & \\ s_{L1}(p, w) & \dots & s_{LL}(p, w) \end{bmatrix}$, where each element is a substitution effect, which measures the differential change of a commodity due solely to a differential change in relative prices when wealth is adjusted so the consumer can just afford his original bundle

$$s_{lk}(p, w) = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w) \tag{20}$$

- $\frac{\partial x_l(p,w)}{\partial p_k} dp_k$ is the resulting change in demand for good l if nominal wealth is unadjusted
- $x_k(p, w)dp_k$ is the nominal wealth adjustment so the consumer can just afford his original bundle. Multiple it by the marginal rate of change $\frac{\partial x_l(p,w)}{\partial w}$ gives the total change in demand for good l due solely to the wealth adjustment
- By definition of $s_{lk}(p, w)$ being only a rate of change, the resulting change (substitution) on good l is $s_{lk}(p, w)dp_k$, which is exactly the sum of the above two terms.

- Proposition 2.F.2 If $x(p, w)$ satisfies differentiability, the Walras' law, homogeneity of degree zero, and the WA, then $\forall(p, w), S(p, w)$ is negative semidefinite.
- Note: If $S(p, w)$ is negative semidefinite and $x(p, w)$ satisfies differentiability, the Walras' law, homogeneity of degree zero, Samuelson (1947) shows that WA is also satisfied if we add a technical condition to $S(p, w)$.

- Negative semidefiniteness implies that $s_{ll}(p,w) \leq 0$, or:

$$\frac{\partial x_l(p,w)}{\partial p_l} + \frac{\partial x_l(p,w)}{\partial w} x_l(p,w) \leq 0 \qquad (21)$$

    - In-class exercise: What can we tell about the relationship between Giffen goods and normal/inferior goods?
- Symmetry
    - When $L = 2$, MWG Ex. 2.F.11 shows that $S(p,w)$ is necessarily symmetric
    - When $L > 2$, $S(p,w)$ need not be symmetric

- In Ch. 3, we will see that symmetry is necessary for generating demand from the maximization of rational preference
  - We cannot directly apply some results in Ch. 1 between WA and rational preference because:
    - Recall that Proposition 1.D.2 says If $\mathcal{B}$, a set of some nonempty subsets of the set of alternatives X, includes all subsets of X of up to three elements, then there is a unique $\succsim$ that "rationalizes" the choice rule.
    - But the Walrasian budget sets does not include all possible subsets of the 2- or 3-commodity bundles
- In Homework MWG Ex. 2.F.2, you are asked to show that the satisfaction of the WA does not imply the preference is rational. This result is due to Hicks (1956).

Proof of Proposition 2.F.2 and the Slutsky Substitution Matrix

- Ex: Show that the compensated law of demand implies that $S(p, w)$ is negative semidefinite.

- Proposition 2.F.3 If $x(p, w)$ satisfies differentiability, the Walras' law, homogeneity of degree zero, and the WA, then $\forall (p, w), p \cdot S(p, w) = 0$ and $S(p, w)p = 0$.
- Proof:
  - MWG Ex. 2.F.2

Other Issues