Social Choice Theory

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Revision History of this File

- 2009/10/1 Clarification. The z in Step 2 is the same as the z in Step 1.
- 2009/9/30 8pm Typo corrected. In Step 5 of the Impossibility theorem proof, it should be $I \setminus S$ is decisive not S is decisive.
- 2009/9/30 8pm We discovered a logical error, which might happen in some cases, in the original definition of IIA in the book. We provided a modified one here.

House Keeping

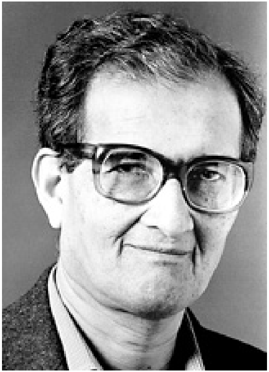
- Grading
  - TAs will do it. The professor will sample some homework everytime.
  - There might be errors in the slides and your graded homeworks. It's your responsibility to alert us. We will deduct points if we find errors in your exams even if you follow the graded materials or slides exactly.
  - If we find that you have the same errors of circulated answers on the Internet of the MWG book and other works, that whole question will get zero. The questions in MWG are well written, we want you to drill it.

$\longrightarrow$Overview

Social Preferences over Two Alternatives
Arrow's Impossibility Theorem
Some Possibility Results

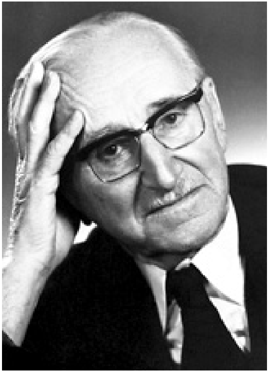
- Welfare Theory
  - “As perhaps the most important of Arrow’s many contributions to welfare theory appears his ‘possibility theorem’, according to which it is impossible to construct a social welfare function out of individual preference functions.” Press Release on 25 October 1972, The Royal Swedish Academy of Sciences.

- Welfare Theory
  - "Sen has clarified the conditions which permit aggregation of individual values into collective decisions, and the conditions which permit rules for collective decision making that are consistent with a sphere of rights for the individual." Press Release on 14 October 1998, The Royal Swedish Academy of Sciences.



- Public Choice Theory
  - “Buchanan’s contribution is that he has transferred the concept of gain derived from mutual exchange between individuals to the realm of political decision-making....Individuals who behave selfishly on markets can hardly behave wholly altruistically in political life. This results in analyses which indicate that political parties or authorities that to at least some extent act out of self-interest, will try to obtain as many votes as possible in order to reach positions of power or receive large budget allocations.” Press Release on 16 October 1986, The Royal Swedish Academy of Sciences.

- Comparative Economic Systems in Information Aggregation
  - "His guiding principle when comparing various systems is to study how efficiently all the knowledge and all the information dispersed among individuals and enterprises is utilized. His conclusion is that only by far-reaching decentralization in a market system with competition and free price-fixing is it possible to make full use of knowledge and information." Press Release on 9 October 1974, The Royal Swedish Academy of Sciences.



Overview

⟶Social Preferences over Two Alternatives

Arrow's Impossibility Theorem

Some Possibility Results

Settings

- Two alternatives: x and y
- Person i's preference over alternatives: α_i
  - $\alpha_i = 1$ if i prefers x
  - $\alpha_i = 0$ if i is indifferent
  - $\alpha_i = -1$ if i prefers y
- I individuals

Settings

- A Social Welfare Functional (SWF) is a rule $F(\alpha_1, \ldots, \alpha_I)$ that assigns a social preference.
  - That is, $F(\alpha_1, \ldots, \alpha_I) \in \{-1, 0, 1\}$ to every possible profile $(\alpha_1, \ldots, \alpha_I) \in \{-1, 0, 1\}^I$.

Examples

- $F(\alpha_1, \ldots, \alpha_I) =$ sign$\sum_i \beta_i \alpha_i$
- Let $(\beta_1, \ldots, \beta_I) \in \mathbb{R}^I_+$
- Question: What's $(\beta_1, \ldots, \beta_I)$ for majority rule? One way to define majority rule is: It is a rule that selects the alternatives that the number of people who like them weakly exceeds the number of people who dislike them.

Definitions

- F is Paretian or has the Pareto property iff $F(1, \ldots, 1) = 1$ and $F(-1, \ldots, -1) = -1$.
- It could be interpreted to capture this meaning: F is Paretian iff you cannot find a new value of F in which one person strictly prefers the new one but everyone else weakly prefers the new one
- Question: Is $F(0, \ldots, 0) = 0$ necessary in a pareto definition so that it is consistent with the above interpretation?

Definitions

- In the book, F is Paretian or has the Pareto property if $F(1, \ldots, 1) = 1$ and $F(-1, \ldots, -1) = -1$.
  - Strictly speaking, this uni-directional (the use of "if" but not with "only if") implication as used in the book for this term is not a definition. A definition should be bidirectional. Be careful, the authors sometimes loosely use only the word "if" instead of "iff" when they seem to want to use for bidirectional statements. At times it is not always very clear what they actually wanted.
  - For this case, I prefer to use "iff" instead of "if" because the authors did use in some proofs the other direction: if F is Paretian, we have $F(1, \ldots, 1) = 1$ and $F(-1, \ldots, -1) = -1$. Let us stick with this one instead.

- F is Dictatorial iff $\exists h$ such that $\alpha_h = x \implies F = x$ where $x \in \{-1, 0, 1\}$.
- Question: F is Paretian $\implies F$ is Dictatorial?

- F is Dictatorial iff $\exists h$ such that $\alpha_h = x \implies F = x$ where $x \in \{-1, 0, 1\}$.
- Question: F is Paretian $\implies F$ is Dictatorial?
  - The answer is no. For the case of two individuals, a function, for example, is Paretian if it satisfies:
    - $F(1,1) = 1, F(0,0) = 0, F(-1,-1) = -1, F(0,1) = 1, F(1,0) = 1$
    - From $F(0,1) = 1, F(1,0) = 1$, we know that either individual is not a dictator.
  - Note that F is Dictatorial $\implies F$ is Paretian (verify it yourself)

Definitions

- F is neutral between alternatives iff the social preference is reversed when we reverse $(\alpha_1, \ldots, \alpha_I)$.
- That is, $\forall(\alpha_1, \ldots, \alpha_I), F(\alpha_1, \ldots, \alpha_I) = -F(-\alpha_1, \ldots, -\alpha_I)$

Math Review: Injective, Surjective and Bijective Functions

Math Review: Injective, Surjective and Bijective Functions

- A function $g : A \rightarrow B$ is injective iff $\forall x, y \in A, f(x) = f(y) \implies x = y$.
  - That is, elements in the range B can only be mapped to at most one element in A.
- A function $g : A \rightarrow B$ is surjective iff $\forall y \in B, \exists x \in A$ such that $y = f(x)$.
  - That is, all elements in the range B are mapped to at least one element in A.
- A function is bijective iff it's both injective and surjective

Math Review: Injective, Surjective and Bijective Functions

- In view of the contradicting definitions of one-to-one (see below), please use the English terms surjective, injective, and bijective. No Chinese terms for these are allowed for exam and homework purposes.
  - Wikipedia.org: "A bijection, or a bijective function is a function f from a set X to a set Y with the property that, for every y in Y, there is exactly one x in X such that $f(x) = y$. Alternatively, f is bijective if it is a one-to-one correspondence between those sets; i.e., both one-to-one (injective) and onto (surjective). (One-to-one function means one-to-one correspondence (i.e., bijection) to some authors, but injection to others.)"

Definitions

- F is anonymous (or symmetric among agents) iff a permutation of preferences across agents does not alter F
- That is, $\forall(\alpha_1, \ldots, \alpha_I), F(\alpha_1, \ldots, \alpha_I) = F(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(I)})$ where
  - $\pi : \{1, \ldots, I\} \rightarrow \{1, \ldots, I\}$ is surjective (i.e. it is an onto function)

Definitions

- F is positively responsive iff x is socially preferred or indifferent to y and some agents raise their consideration of x, then x becomes socially preferred.
- That is, $(\alpha_1, \ldots, \alpha_I) \geq (\alpha'_1, \ldots, \alpha'_I), (\alpha_1, \ldots, \alpha_I) \neq (\alpha'_1, \ldots, \alpha'_I)$ and $F(\alpha'_1, \ldots, \alpha'_I) \geq 0$, then $F(\alpha_1, \ldots, \alpha_I) > 0$.
- Question: What is the difference in meaning if we instead mathematically state that $(\alpha_1, \ldots, \alpha_I) > (\alpha'_1, \ldots, \alpha'_I)$ and $F(\alpha'_1, \ldots, \alpha'_I) \geq 0$, then $F(\alpha_1, \ldots, \alpha_I) > 0$.

Proposition 21.B.1 (May's Theorem)
F is a majority voting SWF $\iff$ it satisfies symmetry among agents, neutrality between alternatives, and positive responsiveness.
Proof:

- "$\Rightarrow$"
  - (Prove it yourself in homework MWG Ex. 21.B.1)
- "$\Leftarrow$"

Proposition 21.B.1 (May's Theorem)
F is a majority voting SWF $\iff$ it satisfies symmetry among agents, neutrality between alternatives, and positive responsiveness.Proof:

- "$\Rightarrow$"
  - (Prove it yourself in homework MWG Ex. 21.B.1)
- "$\Leftarrow$".
  - Denote $n^+(\alpha_1, \ldots, \alpha_I) = \#\{i : \alpha_i = 1\}$ and $n^-(\alpha_1, \ldots, \alpha_I) = \#\{i : \alpha_i = -1\}$. We need to prove three things:
    - $n^+(\alpha_1, \ldots, \alpha_I) = n^-(\alpha_1, \ldots, \alpha_I) \implies F = 0$
    - $n^+(\alpha_1, \ldots, \alpha_I) > n^-(\alpha_1, \ldots, \alpha_I) \implies F > 0$
    - $n^+(\alpha_1, \ldots, \alpha_I) < n^-(\alpha_1, \ldots, \alpha_I) \implies F < 0$

Claim 1: $n^+(\alpha_1, \ldots, \alpha_l) = n^-(\alpha_1, \ldots, \alpha_l) \implies F = 0$

Claim 1: $n^+(\alpha_1, \ldots, \alpha_I) = n^-(\alpha_1, \ldots, \alpha_I) \implies F = 0$

- We will show that $F = -F$, which implies the only number F can give is 0.
- By symmetry among agents, F can be expressed as $F(\alpha_1, \ldots, \alpha_I) = G(n^+(\alpha_1, \ldots, \alpha_I), n^-(\alpha_1, \ldots, \alpha_I))$.
- But $G(n^+(\alpha_1, \ldots, \alpha_I), n^-(\alpha_1, \ldots, \alpha_I))$
  - $= G(n^+(-\alpha_1, \ldots, -\alpha_I), n^-(-\alpha_1, \ldots, -\alpha_I))$ (proof on the next page)
  - $= F(-\alpha_1, \ldots, -\alpha_I)$
  - $= -F(\alpha_1, \ldots, \alpha_I)$ (by neutrality)

(i)

$$n^+(\alpha_1, \ldots, \alpha_l)$$
$$= n^-(\alpha_1, \ldots, \alpha_l) \text{ (given)}$$
$$= n^+(-\alpha_1, \ldots, -\alpha_l) \text{ (by definitions of } n^+ \text{ and } n^-)$$

(ii)

$$n^-(\alpha_1, \ldots, \alpha_l)$$
$$= n^+(\alpha_1, \ldots, \alpha_l) \text{ (given)}$$
$$= n^-(-\alpha_1, \ldots, -\alpha_l) \text{(by definitions of } n^+ \text{ and } n^-)$$

Claim 2: $n^+(\alpha_1, \dots, \alpha_I) > n^-(\alpha_1, \dots, \alpha_I) \implies F > 0$

Proof:

- Define $H = n^+(\alpha_1, \dots, \alpha_I)$ and $J = n^-(\alpha_1, \dots, \alpha_I)$.
- Suppose $n^+(\alpha_1, \dots, \alpha_I) > n^-(\alpha_1, \dots, \alpha_I)$. Then $J < H$.
- WLOG, consider $(\alpha_1, \dots, \alpha_I)$ s.t.
  - $\forall i \leq H : \alpha_i = 1$
  - $\forall i > H : \alpha_i \leq 0$
- Consider a new profile $(\alpha'_1, \dots, \alpha'_I)$ s.t.
  - $\forall i \leq J < H : \alpha'_i = \alpha_i = 1$
  - $\forall i \in (J, H] : \alpha'_i = 0$
  - $\forall i > H : \alpha'_i = \alpha_i \leq 0$
- Then, $n^+(\alpha'_1, \dots, \alpha'_I) = J$ and $n^-(\alpha'_1, \dots, \alpha'_I) = n^-(\alpha_1, \dots, \alpha_I) = J$.
- Hence $F(\alpha'_1, \dots, \alpha'_I) = 0$ by claim 1.
- But $\alpha_{J+1} = 1 > \alpha'_{J+1} = 0$.
- We have $F(\alpha_1, \dots, \alpha_I) = 1$ by positive responsiveness
- Done

Claim 3: $n^+(\alpha_1, \dots, \alpha_I) < n^-(\alpha_1, \dots, \alpha_I) \implies F < 0$
Proof:

- Suppose $n^+(\alpha_1, \dots, \alpha_I) < n^-(\alpha_1, \dots, \alpha_I)$.
- Then $n^-(-\alpha_1, \dots, -\alpha_I) < n^+(-\alpha_1, \dots, -\alpha_I)$ (by definitions of n^+ and n^-)
- $F(-\alpha_1, \dots, -\alpha_I) = 1$ by claim 2
- $-F(-\alpha_1, \dots, -\alpha_I) = F(\alpha_1, \dots, \alpha_I)$ by neutrality
- Hence, $F(\alpha_1, \dots, \alpha_I) = -1$. Done.

- A SWF is $F : \mathcal{A} \to \mathcal{R}$ where:
  - X alternatives
  - $\succsim_i$is rational
  - $\mathcal{R}$ is the set of all possible preference relations on X
  - $\mathcal{P}$ is the set of all possible preference relations on X that no distinct alternatives in X is indifferent
  - $(\succsim_1, \ldots, \succsim_I) \in \mathcal{R}^I$ (Note: $\mathcal{R}$ does not stand for the real line)
  - Define $\mathcal{A} \subseteq \mathcal{R}^{\mathcal{I}}$ as an admissible domain

- Read $xF(\succsim_1, \ldots, \succsim_I)y$ as "x is socially at least as good as y wrt $(\succsim_1, \ldots, \succsim_I)$"
- Let $xF_p(\succsim_1, \ldots, \succsim_I)y$ if $xF(\succsim_1, \ldots, \succsim_I)y$ holds but $yF(\succsim_1, \ldots, \succsim_I)x$ does not
- Read $xF_p(\succsim_1, \ldots, \succsim_I)y$ as "x is socially preferred to y wrt $(\succsim_1, \ldots, \succsim_I)$"

Definition 21.C.2

- $F : \mathcal{A} \rightarrow \mathcal{R}$ is Paretian if $\forall \{x, y\} \subseteq X, \forall (\succsim_1, \ldots, \succsim_I) \in \mathcal{A},$ we have xF_py whenever $\forall i : x \succ_i y$.

Definition 21.C.3 (Modified)

- Please note that the "and" in 21.C.3 in the book (reproduced in the next slide) causes some logical problems. The original conclusion (through negation on $y \succsim_i x \iff y \succsim'_i x$) in class that $x \succ_i y \iff x \succ'_i y$ then $xF_p(\succsim_1, \ldots, \succsim_I)y \iff xF_p(\succsim'_1, \ldots, \succsim'_I)y$ is correct but the derivation causes a logical contradiction (that $x \succ_i y$ and $y \succ_i x$) if we also negate the left hand side ($x \succsim_i y \iff x \succsim'_i y$), which is permitted in the (wrong) definition in the book because of the word "and". I would suggest you to use this one instead:
- Independence of Irrelevant Alternatives (IIA) or Pairwise Independence
  - $F : \mathcal{A} \to \mathcal{R}$ is IIA iff $\forall \{x, y\} \subseteq X, \forall (\succsim_1, \ldots, \succsim_I), (\succsim'_1, \ldots, \succsim'_I) \in \mathcal{A}$, we have: if $\forall i, x \succsim_i y \iff x \succsim'_i y$ then $xF(\succsim_1, \ldots, \succsim_I)y \iff xF(\succsim'_1, \ldots, \succsim'_I)y$, and if $y \succsim_i x \iff y \succsim'_i x$ then $F(\succsim_1, \ldots, \succsim_I)x \iff yF(\succsim'_1, \ldots, \succsim'_I)x$.

Definition 21.C.3

- Independence of Irrelevant Alternatives (IIA) or Pairwise Independence
  - $F : \mathcal{A} \to \mathcal{R}$ is IIA iff $\forall \{x, y\} \subseteq X, \forall (\succsim_1, \dots, \succsim_I), (\succsim'_1, \dots, \succsim'_I) \in \mathcal{A}$, we have: if $\forall i$

$$x \succsim_i y \iff x \succsim'_i y \text{ and } y \succsim_i x \iff y \succsim'_i x$$

then

$$xF(\succsim_1, \dots, \succsim_I)y \iff xF(\succsim'_1, \dots, \succsim'_I)y,$$

and

$$yF(\succsim_1, \dots, \succsim_I)x \iff yF(\succsim'_1, \dots, \succsim'_I)x.$$

Arrow's Impossibility Theorem

- Suppose $\#X \geq 3$, $\mathcal{A} = \mathcal{R}^I$ or $\mathcal{A} = \mathcal{P}^I$. Then every $F : \mathcal{A} \to \mathcal{R}$ that is Paretian and satisfies IIA is dictatorial in this sense:

$$\exists h \text{ s.t.} \{x, y\} \subseteq X, \forall (\succsim_1, \ldots, \succsim_I) \in \mathcal{A}, \text{we have } x F_p y \text{ whenever } x \succ_h y.$$

Arrow's Impossibility Theorem's Classical Proof

- Ten Steps

Arrow's Impossibility Theorem's Classical Proof

- Steps 1-3 show that if a subset of agents is decisive for some pairs of alternatives then it is decisive for all pairs
- Steps 4-6 establish some algebraic properties of the family of decisive subsets
- Steps 7-8 show that there is a smallest decisive set formed by a single agent
- Steps 9-10 prove that this agent is a dictator

Definitions

- Given F, a subset of agents $S \subseteq I$ is:
  - Decisive for x over y if $x \succ_i y, \forall i \in S$ and $y \succ_i x, \forall i \notin S$ then xF_py.
  - Decisive if $\forall \{x, y\} \subseteq X, S$ is decisive for x over y
  - Completely decisive for x over y if $x \succ_i y, \forall i \in S$ then xF_py.

Assumptions
All alternatives in X are distinct hereafter

Step 1

Claim: If $\exists\{x, y\} \subseteq X, S$ is decisive for x over y, then for i) $\forall z \in X$ and $z \neq x, S$ is decisive for x over z. ii) Similarly, for $\forall z \in X$ and $z \neq y, S$ is decisive for z over y.

Proof: We will only show (i). The technique of proofing (ii) is identical.

Step 1 (Part i)

Claim: If $\exists\{x, y\} \subseteq X, S$ is decisive for x over y, then for i) $\forall z \in X$ and $z \neq x, S$ is decisive for x over z.

Proof: First, if $y = z$, a proof in unnecessary. Assume now that $y \neq z$. Consider $(\succsim_1, \ldots, \succsim_I) \in \mathcal{A}$,where:

$$\forall i \in S : x \succ_i y \succ_i z$$
$$\forall i \in I \backslash S : y \succ_i z \succ_i x$$

- It follows that xF_py because S is decisive for x over y.
- $\forall i \in S : y \succ_i z$ and F being Paretian $\implies yF_pz$.
- Since by assumption F is rational, by transitivity, xF_pz.
- By IIA, x is socially preferred to z because
  - no other preference relation, which might imply a different ranking of x, y, z, needs to be considered since it is irrelevant once we've found a preference between x and z
  - indeed, the y we construct above exists because $\mathcal{A}$ is all possible preference relations on X. So it is possible to find a preference relation in which such y exists.
- So S is decisive for x over z

Step 2

Claim: If $\exists\{x, y\} \subseteq X, S$ is decisive for x over y, then $\forall w \in X$, and the z in Step 1, S is decisive for z over w and for w over z.

Proof:

- In-class exercise.

Step 3

Claim: If $\exists \{x, y\} \subseteq X, S$ is decisive for x over y, then S is decisive.

Proof:

Step 3

Claim: If $\exists \{x, y\} \subseteq X, S$ is decisive for x over y, then S is decisive.

Proof:

- Apply step 2 to $\{x, y\}$, we have S is decisive for z over w.
- Apply step 1 to $\{z, w\}$, we have S is decisive for v over w.
- Since $\{v, w\}$ is arbitrary and distinctive, $v \neq w$, done.

Step 4
Claim: If $S, T \subseteq I$ are decisive, then $S \cap T \subseteq I$ are decisive.
Proof:

Step 4

Claim: If $S, T \subseteq I$ are decisive, then $S \cap T \subseteq I$ are decisive.

Proof: Consider $\{x, y, z\} \subseteq X$. By Step 3, we only need to show $S \cap T$ is decisive for x over y.

Consider $(\succsim_1, \ldots, \succsim_I) \in \mathcal{A}$,where:

$$a.\forall i \in S \backslash (S \cap T) : z \succ_i y \succ_i x$$
$$b.\forall i \in S \cap T : x \succ_i z \succ_i y$$
$$c.\forall i \in T \backslash (S \cap T) : y \succ_i x \succ_i z$$
$$d.\forall i \in I \backslash (S \cup T) : y \succ_i z \succ_i x$$

- Then zF_py because S is decisive
- Also, xF_pz because T is decisive
- Since by assumption F is rational, xF_py by transitivity.
- Given $\forall i \in S \cap T : x \succ_i y$ and $\forall i \notin S \cap T : y \succ_i x$ and now x is socially preferred to y (by IIA), by definition, $S \cap T$ is decisive for x over y.
- Done

Step 5

Claim: $\forall S$, either S or $I \backslash S$ (but not both) is decisive.

Proof:

Step 5

Claim: $\forall S$, either S or $I \backslash S$ (but not both) is decisive.

Proof: Consider $\{x, y, z\} \subseteq X$ and $(\succsim_1, \ldots, \succsim_I) \in \mathcal{A}$,where:

$$\forall i \in S : x \succ_i z \succ_i y$$
$$\forall i \in I \backslash S : y \succ_i x \succ_i z$$

- Case 1: xF_py
  - Given $\forall i \in S : x \succ_i y$ and $\forall i \in I \backslash S : y \succ_i x$ and x is socially preferred to y (by IIA), by definition, S is decisive for x over y
  - Hence, by step 3, decisive
- Case 2: yF_px
  - Given that $\forall i : x \succ_i z$, we have xF_pz.
  - By transitivity, yF_pz.
  - By IIA, $I \backslash S$ is decisive for y over z
  - Hence, by step 3, decisive

Step 6
Claim: If S is decisive and $S \subseteq T$, T is also decisive.
Proof:

Step 6

Claim: If S is decisive and $S \subseteq T$, T is also decisive.

Proof:

- By Pareto condition, the empty set of agents $\varnothing$ cannot be decisive.
- Thus $I \backslash T$ cannot be decisive because
  - Otherwise, by step 4, $S \cap I \backslash T = \varnothing$ is decisive.
- Hence, T is decisive by step 5.
- Done.

Step 7

Claim: If S is decisive and $\#S > 1$, then $\exists S' \subseteq S$ and $S' \neq S$ s.t. S' is decisive.

Proof:

Step 7

Claim: If S is decisive and $\#S > 1$, then $\exists S' \subseteq S$ and $S' \neq S$ s.t. S' is decisive.

Proof:

- Take any $h \in S$. If $S \backslash \{h\}$ is decisive, let $S \backslash \{h\} = S'$, done.
- Else if $S \backslash \{h\}$ is not decisive,
  - By step 5, $I \backslash (S \backslash \{h\}) = (I \backslash S) \cup \{h\}$ is decisive.
  - By step 4, $S \cap (I \backslash S) \cup \{h\} = \{h\}$ is decisive
  - Let $\{h\} = S'$
  - Done

Step 8

Claim: $\exists h \in I$ s.t. $\{h\}$ is decisive.

Proof:

Step 8

Claim: $\exists h \in I$ s.t. $\{h\}$ is decisive.

Proof:

- By Pareto condition, I is decisive
- Let $S = I$ and apply step 7
- Iterate step 7 with a smaller S' each time
- Since I is finite, when there are two elements in S, apply step 7 one last time yielding $S' = \{h\}$
- Done

Step 9
Claim: If S is decisive then, $\forall\{x, y\} \subseteq X, S$ is completely decisive for x over y.
Proof:

Step 9

Claim: If S is decisive then, $\forall\{x,y\} \subseteq X, S$ is completely decisive for x over y.

Proof:

- Consider $\{x,y,z\} \subseteq X$ and $(\succsim_1, \ldots, \succsim_I) \in \mathcal{A}$,where:

$$a.\forall i \in S : x \succ_i z \succ_i y$$
$$b.\forall i \in T : x \succ_i y \succ_i z$$
$$c.\forall i \in I \backslash (S \cup T) : y \succ_i z \succ_i x$$

- xF_pz because by step 6 $S \cup T$ is decisive since $S \subseteq S \cup T$ and S is decisive
- zF_py because S is decisive
- Since by assumption F is rational, xF_py by transitivity
- By IIA, x is socially preferred to y
- So S is completely decisive for x over y
- Done

Step 10
Claim: If $\exists h$ s.t. $S = \{h\}$ is decisive, then h is a dictator.
Proof:

Step 10

Claim: If $\exists h$ s.t. $S = \{h\}$ is decisive, then h is a dictator.

Proof:

- If $\{h\}$ is decisive, then $\forall \{x, y\} \subseteq X$, we have h is completely decisive for x over y.
- S being singleton and completely decisive is by definition the same as h being a dictator.
- Q.E.D.

Arrow's Impossibility Theorem's Classical Proof: Recap

- Steps 1-3 show that if a subset of agents is decisive for some pairs of alternatives then it is decisive for all pairs
- Steps 4-6 establish some algebraic properties of the family of decisive subsets
- Steps 7-8 show that there is a smallest decisive set formed by a single agent
- Steps 9-10 prove that this agent is a dictator

- Possibility results can be obtained using:
  - irrational social preference
  - restricted domain

- Last remarks on the relations between social preference and social choice function wrt Arrow's Impossibility Theorem